where $g$ and $T$ are defined as

$$\partial f/\partial \beta = g \tag{5}$$

$$\partial S/\partial \beta = T \tag{6}$$

Equations (3) are a set of simultaneous linear ordinary differential equations for $g$ and $T$. The functions $f$ and $S$ and their derivatives appear as coefficients. Once Eqs. (3) under the boundary conditions (4) are solved for $g$ and $T$ the required solutions for $f$ and $S$ can then be obtained from the first-order differential Eqs (5) and (6) either by a simple quadrature or using a Runge-Kutta procedure.

Equations (3) and (4) are solved by the following procedure of reducing the boundary value problem to an initial value problem. Let $g$ and $T$ be denoted by

$$g = g_1 + \lambda g_2 \tag{7}$$

$$T = T_1 + \mu T_2 \tag{8}$$

so that the required initial conditions $g''(0)$ and $T'(0)$ now become $\lambda$ and $\mu$, respectively. These are subsequently determined from the given conditions at the other end viz $g'(\infty) = 0$ and $T(\infty) = 0$

In view of the Eqs. (7) and (8) Eqs. (3) and (4) simplify to

$$g_1''' + fg_1'' + f''g_1 - 2\beta f'g_1' = f'^2 - S - \beta T \tag{9}$$

$$g_2''' + fg_2'' + f''g_2 - 2\beta f'g_2' = 0 \tag{10}$$

with

$$g_1(0) = g_1'(0) = g_1''(0) = 0, \tag{11}$$

$$g_2(0) = g_2'(0) = 0, \quad g_2''(0) = 1 \tag{12}$$

and

$$T_1'' + fT_1' = -gS' \tag{13}$$

$$T_2'' + fT_2' = 0 \tag{14}$$

with

$$T_1(0) = T_1'(0) = 0 \tag{15}$$

$$T_2(0) = 0, \quad T_2'(0) = 1 \tag{16}$$

For integration of Eqs. (9) to (16) the two-, three-, and four-point Falkner's predictor and Adam's corrector formulas[7] are employed. The step length in $\eta$ for integration across the boundary layer is taken to be 0.1. and the integration is carried up to $\eta = 9$, taken to be the edge of the boundary layer.

In starting the solution of Eqs. (9) to (16) the Blasius value of $f''(0) = 0.4695999$ is assumed. Hence at $\beta = 0$, $S'(0)$ is also known. To obtain the solutions of $g$ and $T$ at a small step $\Delta\beta$, the values of functions $f(\eta)$, $S(\eta)$ and their derivatives at this $\beta$ are required. The following iterative procedure was employed for this purpose.

As the distributions of $f(\eta)$ and $S(\eta)$ and their derivatives, are known at the previous $\beta$ (to start with, for a given $Sw$, $\beta = 0$ and the solutions $f(\eta)$, $S(\eta)$ were obtained from Eqs. (1) and (2)), they are used as first approximations in Eqs. (9) to (16) for the current $\beta$ and the solutions for $g(\eta)$ and $T(\eta)$ are obtained. These are then used to obtain $f(\eta)$ and $S(\eta)$ and their derivatives which serve as second approximations, again for use in Eqs. (9) to (16). This procedure is repeated till the final distributions for $f(\eta)$ and $S(\eta)$ settle down. In the present work the iteration is stopped when the sum

$$|f''_{(j+1)}(0) - f''_j(0)| + |S'_{(j+1)}(0) - S'_j(0)|,$$

where the subscripts $j$ and $(j+1)$ denote the $j$ th and $(j+1)$ th iteration, is less than $10^{-7}$.

As the purpose of the present Note is to demostrate the effectiveness of the method for the solution of the boundary-layer Eqs. (1) and (2), the solutions are obtained only for a few representative values of $Sw$ and $\beta$. The results are compared with those of Cohen and Reshotko[8] in the Table 1. These authors obtained their solutions by the method of successive approximations reducing the Eqs. (1) and (2) to a set of integral equations. The comparison between the present method and that of Cohen and Reshotko[8] shows that the methods give identical values correct to 3 decimal places. The authors of that reference have themselves mentioned that their work is correct to $\pm$ 0.0002.

**Table 1 Wall shear and heat-transfer parameters**

| $S_w$ | $\beta$ | f″(0) Present method | f″(0) Cohen and Reshotko | S′(0) Present method | S′(0) Cohen and Reshotko |
|---|---|---|---|---|---|
| | −0.3 | 0.3178 | 0.3182 | 0.4261 | 0.4262 |
| | −0.2 | 0.3874 | . . . | 0.4477 | . . . |
| | −0.15 | 0.4125 | . . . | 0.4547 | . . . |
| | −0.14 | . . . | 0.4166 | . . . | 0.4554 |
| 0.0 | −0.1 | 0.4340 | . . . | 0.4605 | . . . |
| | 0.0 | 0.4696 | 0.4696 | 0.4696 | 0.4696 |
| | 0.5 | 0.5812 | 0.5806 | 0.4942 | 0.4948 |
| | 2.0 | 0.7387 | 0.7381 | 0.5206 | 0.5203 |
| | −0.14 | . . . | 0.3841 | . . . | 0.359 |
| | −0.125 | 0.3954 | . . . | 0.3617 | . . . |
| | −0.100 | 0.4122 | . . . | 0.3650 | . . . |
| | −0.075 | 0.4278 | . . . | 0.3680 | . . . |
| | −0.050 | 0.4425 | . . . | 0.3708 | . . . |
| 0.2 | −0.025 | 0.4564 | . . . | 0.3733 | . . . |
| | 0.0 | 0.4696 | 0.4696 | 0.3757 | 0.3757 |
| | 0.5 | 0.6546 | 0.6547 | 0.4036 | 0.4030 |
| | 1.5 | 0.8695 | 0.8689 | 0.4267 | 0.4261 |
| | 2.0 | 0.9483 | 0.9480 | 0.4334 | 0.4331 |

### References

[1] Rubbert, P. E., "Analysis of Transonic Flow by Means of Parametric Differentiation," Ph. D. dissertation, 1965, MIT, Cambridge, Mass.

[2] Rubbert, P. E. and Landahl, M. T. "Solution of the Transonic Airfoil Problem through Parametric Differentiation," AIAA Paper 66-90, New York, 1966.

[3] Landahl, M. T., *Unsteady Transonic Flow*, Pergamon Press, New York, 1961.

[4] Landahl, M. T., "Linearised Theory of Unsteady Transonic Flow," *Symposium Transonicum*, Springer-Verlag, Berlin, 1964.

[5] Rubbert, P.E. and Landahl, M. T., "Solution of Nonlinear Flow Problems through Parametric Differentiation," *The Physics of Fluids*, Vol. 10, No. 4, April 1967, pp. 831–835.

[6] Stewartson, K., "The Theory of Laminar Boundary Layers in Compressible Fluids," *Oxford Mathematics Monographs*, 1964.

[7] Smith, A. M. O. and Clutter, D. W., "Solution of Prandtl's Boundary Layer Equations," Engineering Paper 1530, Nov. 1962, Douglas Aircraft Co., Long Beach, Calif.

[8] Cohen, C. B. and Reshotko, E., "Similar Solutions for the Compressible Laminar Boundary Layer with Heat Transfer and Pressure Gradient," TR-1293, 1956, NACA.

# Rayleigh Wave Effects in an Elastic Half-Space

H. R. AGGARWAL*
*Space Science Division, NASA Ames Research Center, Moffett Field, Calif.*

THE importance of the Rayleigh waves since first demonstrated by Lamb[1] is unquestioned in dynamic half-space problems. Many investigations following Lamb's work have, as a part of the study, looked for their effects. Asymptotic expressions for the Rayleigh phase have been obtained in some recent works.[2–4] As is well known, all these studies are carried out on the basis of the far-field assumption. The concluding remark by Rayleigh[5]

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* Senior Postdoctoral National Research Council Associate.

"...they must acquire *at a great distance from the source* a continually increasing preponderance" and the alike phrases met with in discussions relating to Rayleigh waves tend to give an impression that these waves occur in the far field only. A remark or an example showing that Rayleigh waves may, under certain loadings, also occur in the near field does not seem to appear anywhere in literature. Given in this Note is an example wherein Rayleigh waves also occur in the near field. Information such as this may be helpful in understanding of the mounds and ridges observed at the center of some lunar craters. It may also explain, in part, the failure of the rear sheet in a bumper-hull system of a spaceship when hit by a meteoroid.

Consider a homogeneous isotropic linearly elastic half-space subject to a uniform disk pressure loading with Dirac's delta function, $\delta(t/t_0)$, time dependence where $t_0$ is some fixed time introduced for proper nondimensionalization of the results to follow. The Laplace transforms of the radial and the vertical displacements $u_r$ and $u_z$, respectively, for this loading are obtained in Ref. 6 [Eqs. (12)] and may be written as

$$\bar{u}_r(r,z,p) = \frac{\sigma_0 t_0 a}{\mu}\int_0^\infty \frac{xJ_1(krx)J_1(kax)}{M(x)} \times [(2x^2+1)e^{-k\alpha z} - 2\alpha\beta e^{-k\beta z}]\,dx$$

$$\bar{u}_z(r,z,p) = \frac{\sigma_0 t_0 a}{\mu}\int_0^\infty \frac{J_0(krx)J_1(kax)}{M(x)} \times \alpha[(2x^2+1)e^{-k\alpha z} - 2x^2 e^{-k\beta z}]\,dx \qquad (1)$$

where $p$ is the Laplace transform parameter $\sigma_0 = -P\ (>0)$, $P$ being the pressure, $a$ is the radius of the load-disk, $\mu$ is the shear modulus, $J_0$ and $J_1$ are the Bessel functions of the first kind of orders 0 and 1, respectively,

$$k = p/c_2, \quad \alpha = (x^2+a_1^2)^{1/2}, \quad \beta = (x^2+1)^{1/2}, \quad a_1 = c_2/c_1$$
$$M(x) = (2x^2+1)^2 - 4x^2\alpha\beta$$

and $c_1$ and $c_2$ are the dilatational and the shear wave velocities. The radicals $\alpha$ and $\beta$ are assigned the branch which yields a positive real part.

At the axis of symmetry, $r = 0$, Eqs. (1) reduce to

$$\bar{u}_r(0,z,p) = 0$$

$$\bar{u}_z(0,z,p) = \frac{\sigma_0 t_0 a}{\mu}\int_0^\infty \frac{x^2 J_1(kax)}{M(x)}[F_c(x)e^{-k\alpha z} - F_s(x)e^{-k\beta z}]\,dx \qquad (2)$$

where

$$F_c(x) = \alpha(2x^2+1)/x^2, \qquad F_s(x) = 2\alpha.$$

Inverse Laplace transform of wave integrals of the type

$$\bar{I}_n(a,z,p) = \int_0^\infty x^{n+1}J_n(kax)f(x)e^{-kz(x^2+a_j^2)^{1/2}}\,dx, \quad n = 0,1,2,\ldots \qquad (3)$$

was obtained in Ref. 4 achieved in a simple way through the use of the extended Bateman-Pekeris theorem.[7] In Eq. (3), $a_j$ is a parameter equal to $a_1(=c_2/c_1)$ for a dilatational wave and 1 for a shear wave, $f(x)$ is a rational function of $x^2$, $\alpha$ and $\beta$. It is shown in Ref. 4 that the path of integration passes through the Rayleigh pole,[4,8] $x = \imath\gamma$, given by $M(\imath\gamma) = 0$. For shallow depths, $z \ll R$, $R^2 = a^2+z^2$, this pole lies close to the subsequent deformed path of integration and, therefore, has significant influence on the value of the integral. Expansion of the integrand in its neighborhood yields an approximate formula for the Rayleigh wave effects, obtained as[4]

$$I_n \approx -[2^{1/2}c_2\gamma^{n+1/2}/\dot{M}(\imath\gamma)(az)^{1/2}]\,\mathrm{Re}\,[F(\imath\gamma)Z_{a_j}^{-1/2}] \qquad (4)$$

where the function $F(x)$ is defined by $f(x) = F(x)/M(x)$,

$$Z_{a_j} = (\tau-\gamma)/l - \imath(\gamma^2-a_j)^{1/2}, \quad -\pi \leqq \arg Z_{a_j} < \pi, \quad \tau = c_2 t/a,$$
$$l = z/a \ (l \ll 1), \quad \dot{M}(\imath\gamma) = (dM/dx)_{x=\imath\gamma}$$

and Re implies the real part.

At points on the surface $l = 0$ Eq. (4) simplifies to

$$I_n \approx -[2^{1/2}c_2\gamma^{n+1/2}/\dot{M}(\imath\gamma)a]\,\mathrm{Re}\,[F(\imath\gamma)/(\tau-\gamma)^{1/2}] \qquad (5)$$

From the definition of $\gamma$ and Eq. (1) in Ref. 8, it follows that $\gamma = c_2/c_R$ where $c_R$ is the Rayleigh wave velocity. The instant $\tau = \gamma$, thus, marks the arrival time of the Rayleigh wave.

In the present case, the Rayleigh waves near the center of loading arise from the portion of the dilatational and shear waves moving towards the axis, after they originate at the edge of the load-disk. The functions $F_c(x)/M(x)$ and $F_s(x)/M(x)$ together with their exponential factors satisfy the order conditions of the extended Bateman-Pekeris theorem, and the results in Eqs. (4) and (5) can, therefore, be applied repeatedly to the integral in Eq. (2). This gives the Rayleigh phase for $u_z$ at $r = 0$ as†

$$u_z|_{R.P.} \approx \frac{2^{1/2}\sigma_0 t_0 c_2(\gamma^2-a_1^2)^{1/2}}{\mu\dot{M}(\imath\gamma)\gamma^{1/2}}\left(\frac{a}{z}\right)^{1/2} \times \mathrm{Im}[(2\gamma^2-1)Z_{a_1}^{-1/2} - 2\gamma^2 Z_1^{-1/2}], \quad z \neq 0 \qquad (6a)$$

$$\approx \frac{2^{1/2}\sigma_0 t_0 c_2(\gamma^2-a_1^2)^{1/2}}{\mu\dot{M}(\imath\gamma)\gamma^{1/2}}\frac{1}{(\gamma-\tau)^{1/2}}, \quad \tau < \gamma, \quad z = 0 \qquad (6b)$$

where Im denotes the imaginary part.

For Poisson's ratio $\nu = 0.25$, $a_1 = 1/(3)^{1/2}$, $\gamma^2 = (3+(3)^{1/2})/4$, $\dot{M}(\imath\gamma) = -8\gamma/(3)^{1/2}$. A plot of Eq. (6a) giving the effect of Rayleigh wave on the vertical displacement for $\nu = 0.25$ at times near $\tau \doteq \gamma$ is exhibited in Fig. 1. It shows that the influence of the Rayleigh wave increases with $z$ decreasing which, by Eq. (6b), becomes infinite like $(\gamma-\tau)^{-1/2}$ at $z = 0$. One notes that, for a given $z$, the maximum effect occurs at a time shortly before the arrival time of the Rayleigh wave, given by $\tau \approx \gamma - 0.2l$.

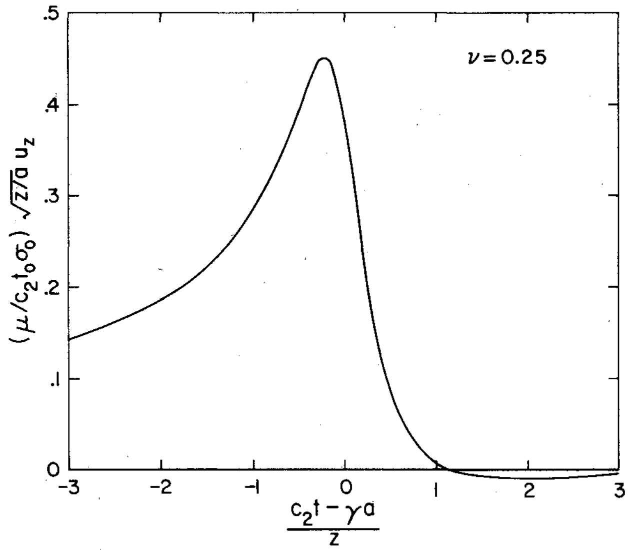


**Fig. 1 Vertical displacement due to Rayleigh waves at points on the axis near the surface of an elastic half-space subjected to an impulsive uniform disc pressure loading.**

## References

[1] Lamb, H., "On the Propagation of Tremors over the Surface of an Elastic Solid," *Philosophical Transactions of the Royal Society of London*, Vol. 203, Series A, 1904, pp. 1–42.

[2] Perkeris, C. L. and Lifson, H., "Motion of the Surface of a Uniform Elastic Half-Space Produced by a Buried Pulse," *Journal of the Acoustical Society of America*, Vol. 29, No. 11, 1957, pp. 1233–1238.

[3] Chao, C. C., Bleich, H. H., and Sackman, J., "Surface Waves in an Elastic Half-Space," *Journal of Applied Mechanics*, Vol. 28, 1961, pp. 300–301.

† The far-field Rayleigh wave approximation may, as in Ref. 9, be obtained by a suitable integration in space of the corresponding result for an impulsive surface normal point load.

[4] Aggarwal, H. R. and Ablow, C. M., "Solution to a Class of Three-Dimensional Pulse Propagation Problems in an Elastic Half-Space," *International Journal of the Engineering Sciences*, Vol. 5, 1967, pp. 663–679.

[5] Lord Rayleigh, "On Waves Propagated Along the Plane Surface of an Elastic Solid," *Proceedings of the London Mathematical Society*, Vol. 17, 1885, pp. 4–11.

[6] Mitra, M., "Disturbance Produced in an Elastic Half-Space by Impulsive Normal Pressure," *Proceedings of the Cambridge Philosophical Society*, Vol. 60, 1964, p. 685.

[7] Aggarwal, H. R. and Ablow, C. M., "Extension of the Bateman-Pekeris Theorem with Application to a Wave Propagation Problem," *Journal of Mathematics and Physics*, Vol. 44, 1965, pp. 267–270.

[8] Aggarwal, H. R. and Ablow, C. M., "On the Rayleigh Equation for Elastic Surface Waves," *Journal of the Acoustical Society of America*, Vol. 47, Part 2, 1970, p. 1461.

[9] Baron, M. L. and Check, R., "Elastic Rayleigh Wave Motions Due to Nuclear Blasts," *Transactions of the ASCE: Journal of the Engineering Mechanics Division*, Vol. 89, No. EM1, 1963, pp. 57–70.

# Strongly Coupled Stress Waves in Heterogeneous Plates

A. S. D. Wang,* P. C. Chou,† and J. L. Rose ‡
*Drexel University, Philadelphia, Pa.*

## Introduction

In the analysis of heterogeneous anisotropic plates,[1,2] the general equations governing the linear and the rotatory motions of the plate may be expressed by a set of five simultaneous second order partial differential equations. Depending on the heterogeneity and the anisotropy of the plate, strong coupling amongst the displacement functions and their derivatives may exist both in the field equations and in the initial/boundary conditions. A general solution to such a system seems improbable, although a few simplified problems concerning statical or quasi-statical deformations have been solved.

In the present Note, we consider coupled stress waves which are generated by an impulsive load applied on the one end of a semi-infinite plate. The field equations governing the one dimensional coupled waves are readily obtained from the general theory in Refs. 1 or 2. A general form of these equations may be expressed by

$$u_{1,xx} - c_1^{-2}u_{1,tt} + A_1u_{2,xx} = R_1 \quad (1a)$$

$$u_{2,xx} - c_2^{-2}u_{2,tt} + A_2u_{1,xx} = R_2 \quad (1b)$$

$$u_{3,xx} - c_3^{-2}u_{3,tt} = R_3 \quad (1c)$$

where $u_i$, $i = 1, 2, 3$, denote the generalized displacements; $c_i$, $A_1$, and $A_2$ may be real, continuous functions of $x$; and

$$R_i = \sum_{j=1}^{3} (\alpha_{ij}u_j + \beta_{ij}u_{j,x}) \qquad i = 1, 2, 3 \quad (2)$$

with $\alpha_{ij}$ and $\beta_{ij}$ being at most real continuous functions of $x$.

If in Eq. (1), $A_1 = A_2 = 0$, the system reduces to what is termed as "weakly" coupled hyperbolic system for which a unified solution scheme by the method of characteristics has been presented in Ref. 3. In the present problem however, with $A_1 \neq A_2 \neq 0$, a strong coupling in the second derivatives exists. We shall extend the method described in Ref. 3 and treat the transient stress waves in a semi-infinite plate, subjected to an initial step input. Coupled discontinuity fronts are found to propagate at different velocities. The normal plate stress and the bending moment at different time regimes are illustrated through graphs.

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* Associate Professor of Applied Mechanics.
† Professor of Aerospace Engineering.
‡ Assistant Professor of Mechanical Engineering.

### The Method of Characteristics

Following a similar procedure as outlined in Ref. 3, § the six characteristics of the system of equations (1) are found to be

$$\frac{dx}{dt} = \begin{cases} \pm\{[(1+s)/2](c_1^2 + c_2^2)\}^{1/2} = \pm s_1 \\ \pm\{[(1-s)/2](c_1^2 + c_2^2)\}^{1/2} = \pm s_2 \\ \pm c_3 \end{cases} \quad (3)$$

where

$$s = \{1 - 4c_1^2c_2^2(1 - A_1A_2)/(c_1^2 + c_2^2)^2\}^{1/2} \quad (4)$$

If all of the six characteristics are real and distinct, the system (1) is said to be totally hyperbolic,[5] or distinctly hyperbolic.[6] In what follows, we shall limit our discussions to such a distinctly hyperbolic system. Then, the following inequality must be satisfied everywhere in the physical plane:

$$-(c_1^2 - c_2^2)^2/4c_1^2c_2^2 < A_1A_2 < 1 \quad (5)$$

The six independent equations associated with the six characteristics are given by

$$I_1^{\pm}: \quad (c_1^2R_1dx - s_1^2du_{1,x} \pm s_1du_{1,t})(c_2^2 - s_1^2) - A_1c_1^2(c_2^2R_2dx - s_1^2du_{2,x} \pm s_1du_{2,t}) = 0 \quad (6)$$

$$I_2^{\pm}: \quad (c_2^2R_2dx - s_2^2du_{2,x} \pm s_2du_{2,t})(c_1^2 - s_2^2) - A_2c_2^2(c_1^2R_1dx - s_2^2du_{1,x} \pm s_2du_{2,t}) = 0 \quad (7)$$

$$I_3^{\pm}: \quad R_3dx - du_{3,x} \pm (1/c_3)u_{3,t} = 0 \quad (8)$$

Equations $I_1^{\pm}$ govern the variables $u_i$ and their first derivatives along, respectively, curves $C_1^{\pm}$ whose local slopes are respectively $\pm s_1$; similarly, equations $I_2^{\pm}$ are for curves $C_2^{\pm}$ whose slopes are $\pm s_2$; and equations $I_3^{\pm}$ are for curves $C_3^{\pm}$ with local slopes of respectively $\pm s_3$.

If in a region in the physical plane the first derivatives of $u_1$ are continuous, we have the following additional relations,

$$du_i = u_{i,x}dx + u_{i,t}dt \qquad i = 1, 2, 3. \quad (9)$$

The nine equations (6–9) will enable the determination of the nine variables $u_i$, $u_{i,x}$ and $u_{i,t}$ $(i = 1, 2, 3)$, if proper initial and boundary conditions are prescribed. But, when discontinuities in the first derivatives of $u_i$ exist, equations governing their magnitudes must be used in place of Eq. (9). Following a similar but somewhat complicated procedure (for details see Ref. 4) as that outlined in Ref. 3, we can show that discontinuities in the first derivatives of $u_i$ may occur only across a characteristic curve; and for the system of equations (6–8) in particular, discontinuities in $u_{j,x}$ and $u_{j,t}$, $(j = 1, 2)$, may occur only across $C_j^{\pm}$; discontinuities in $u_{3,x}$ and $u_{3,t}$ may occur only across $C_3^{\pm}$. The equations that govern these discontuities are given by,

$$[u_{i,x}]_{C_i^{\pm}} = \mp(1/s_i)[u_{i,t}]_{C_i^{\pm}} = K_i \exp\{\textstyle\int Q_i dx\} \qquad i = 1, 2 \quad (10)$$

$$[u_{j,x}]_{C_i^{\pm}} = \mp(1/s_i)[u_{j,t}]_{C_i^{\pm}} = [A_jc_j^2K_i/(c_i^2 - s_j^2)]\exp\{\textstyle\int Q_i dx\} \qquad i \neq j; \quad i,j = 1, 2 \quad (11)$$

$$[u_{3,x}]_{C_3^{\pm}} = \mp(1/c_3)[u_{3,t}]_{C_3^{\pm}} = K_3 \exp\{\textstyle\int (\beta_{33}/2)dx\} \quad (12)$$

where

$$Q_i = \frac{A_i\beta_{ji} - \left(1 - \dfrac{s_i^2}{c_j^2}\right)\beta_{ii} + \dfrac{A_iA_jc_j^2\beta_{jj}}{c_i^2 - s_j^2} - \left(1 - \dfrac{s_i^2}{c_j^2}\right)\dfrac{\beta_{ij}A_jc_j^2}{c_i^2 - s_j^2}}{2\{s_i^2A_iA_j/(c_i^2 - s_j^2) - s_i^2(c_j^2 - s_i^2)/c_i^2c_j^2\}} \qquad i \neq j; \quad i,j = 1, 2 \quad (13)$$

The notation $[f]_C$ represents the magnitude of discontinuity of the function $f$ across the characteristic curve $C$. The constants $K_i$, $i = 1, 2, 3$, must be determined from appropriate initial and

§ Detail formulations of the characteristic determinant, the characteristic equations, the jump conditions and the numerical procedures are contained in a separate report.[4]